CES functional forms

A digression

A digression on CES functions

- Production function
  - National differentiation of inputs
  - Firm level differentiation
  - Capital-labor substitution
- Consumer demand
  - National differentiation of goods
  - Specialized goods
  - Upper-tier preferences

CES production functions

- Assume the following:

$$y = \left[\sum_i \alpha_i x_i^{\rho} \right]^{1/\rho} \quad 1 > \rho > 0$$

CES production functions

- From first order conditions:

$$\ell = \sum_i \omega_i x_i + \lambda \left(y - \left[\sum_i \alpha_i x_i^{\rho} \right]^{1/\rho} \right)$$

$$\frac{d\ell}{dx_i} = \omega_i - \lambda \left[\sum_j \alpha_j x_j^{\rho} \right]^{\frac{1}{\rho}-1} \alpha_i x_i^{\rho-1} = 0$$

$$\frac{d\ell}{d\lambda} = y - \left[\sum_i \alpha_i x_i^{\rho} \right]^{1/\rho} = 0$$

CES production functions

- Rearranging:

$$\lambda = \omega_i \left[\sum_j \alpha_j x_j^{\rho} \right]^{1-\frac{1}{\rho}} \alpha_i^{\ \rho-1} x_i^{1-\rho}$$

- And since λ=λ

$$x_j = x_i \left(\frac{\alpha_j}{\alpha_i} \right)^{\frac{1}{1-\rho}} \left(\frac{\omega_i}{\omega_j} \right)^{\frac{1}{1-\rho}}$$

CES production functions

$$x_j = x_i \left(\frac{\alpha_j}{\alpha_i}\right)^{\frac{1}{1-\rho}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{1}{1-\rho}}$$

- We shall now represent the elasticity of substitution by the term σ. It should be clear from the equation above that the (Allen) substitution elasticity between two inputs x_i and x_j will be σ=1/(1-ρ). Hence, we can simplify equation as follows:

$$x_j = x_i \left(\frac{\alpha_j}{\alpha_i}\right)^{\sigma} \left(\frac{\omega_i}{\omega_j}\right)^{\sigma}$$

CES production functions

$$x_j = x_i\left(\frac{\alpha_j}{\alpha_i}\right)^{\sigma}\left(\frac{\omega_i}{\omega_j}\right)^{\sigma}$$

$$\frac{d\ell}{d\lambda} = y - \left[\sum_i \alpha_i x_i^{\rho}\right]^{1/\rho} = 0$$

- We make some substitutions

$$y = \left[x_i^{\rho}\sum_j \alpha_j \left(\frac{\alpha_j}{\alpha_i}\right)^{\rho\sigma}\left(\frac{\omega_i}{\omega_j}\right)^{\rho\sigma}\right]^{1/\rho}$$

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$$y = \left[x_i^{\rho} \sum_j \alpha_j \left(\frac{\alpha_j}{\alpha_i} \right)^{\rho\sigma} \left(\frac{\omega_i}{\omega_j} \right)^{\rho\sigma} \right]^{1/\rho}$$

- Collecting terms

$$y = \left[x_i^{\rho} \alpha_i^{-\rho\sigma} \omega_i^{\rho\sigma} \sum_j \alpha_j^{1+\rho\sigma} \omega_j^{-\rho\sigma} \right]^{1/\rho}$$

CES production functions

- Next we manipulate the definition of σ, which yields (1+ρσ)=σ and hence -ρσ=1-σ. We also have 1/ρ=- σ/(1-σ) and 1-(1/ρ)=1/(1-σ). Making these substitutions our last equation then yields the following:

$$y = x_i \left(\frac{\omega_i}{\alpha_i} \right)^{\sigma} P^{-\sigma}$$

CES price index

$$P = \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma} \right]^{1-\frac{1}{\rho}} = \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma} \right]^{1/(1-\sigma)}$$

$$x_i = Y \left(\frac{\alpha_i}{\omega_i} \right)^{\sigma} P^{\sigma - 1}$$

CES production functions

- Relative input demands are represented by

$$x_j = x_i \left(\frac{\alpha_j}{\alpha_i}\right)^{\frac{1}{1-\rho}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{1}{1-\rho}}$$

- Total input demand given total output is represented by equation

$$x_i = y\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma} P^{\sigma}$$

- Total input demand given total cost is given by equation

$$x_i = Y\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma} P^{\sigma-1}$$

- The input values as we move along an isoquant are given by

$$x_i = y\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma} P^{\sigma}$$

- The unit cost function is scale independent, and is represented by *P* in equation

$$c = \sum_i \omega_i \left(\frac{\alpha_i}{\omega_i}\right)^{\sigma} P^{\sigma} = P^{\sigma} \sum_i \alpha_i^{\sigma} \omega_i^{1-\sigma} = P^{\sigma} \left(\sum_i \alpha_i^{\sigma} \omega_i^{1-\sigma}\right)^{\frac{1-\sigma}{1-\sigma}} = P^{1+\sigma-\sigma} = P$$

$$P = \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma}\right]^{1-\frac{1}{\rho}} = \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma}\right]^{1/(1-\sigma)}$$

CES consumption functions

- Relative final demands are represented by

$$x_j = x_i \left(\frac{\alpha_j}{\alpha_i}\right)^{\frac{1}{1-\rho}} \left(\frac{\omega_i}{\omega_j}\right)^{\frac{1}{1-\rho}}$$

- Demand for a good given total utility is represented by equation

$$x_i = y\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma} P^{\sigma}$$

- Demand given total income is given by equation

$$x_i = Y\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma} P^{\sigma-1}$$

- Quantities as we move along an indifference curve are given by

$$x_i = y\left(\frac{\alpha_i}{\omega_i}\right)^{\sigma} P^{\sigma}$$

- The CPI is scale independent, and is represented by *P* below

$$c = \sum_i \omega_i \left(\frac{\alpha_i}{\omega_i}\right)^{\sigma} P^{\sigma} = P^{\sigma} \sum_i \alpha_i^{\sigma} \omega_i^{1-\sigma} = P^{\sigma} \left(\sum_i \alpha_i^{\sigma} \omega_i^{1-\sigma}\right)^{\frac{1-\sigma}{1-\sigma}} = P^{1+\sigma-\sigma} = P$$

$$P = \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma}\right]^{1-\frac{1}{\rho}} = \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma}\right]^{1/(1-\sigma)}$$