Notes on CES production functions

Consider a production function of the following form:

$$(1)\quad y=\left[\sum_i \alpha_i x_i^{\rho}\right]^{1/\rho} \quad 1>\rho>0$$

How can we characterize demand for inputs in this case? We start by assuming that firms seek to maximize profits from producing y subject to input prices ω. This will involve unit cost minimization. Formally, we solve for unit cost minization as a constrained optimization problem. We first set up the following cost-minimizing Lagrangian in equation (2):

$$(2)\quad \ell=\sum_i \omega_i x_i+\lambda\left(y-\left[\sum_i \alpha_i x_i^{\rho}\right]^{1/\rho}\right)$$

The first order conditions are as follows:

$$(3)\quad \frac{d\ell}{dx_i}=\omega_i-\lambda\left[\sum_j \alpha_j x_j^{\rho}\right]^{\frac{1}{\rho}-1}\alpha_i x_i^{\rho-1}=0$$

$$(4)\quad \frac{d\ell}{d\lambda}=y-\left[\sum_i \alpha_i x_i^{\rho}\right]^{1/\rho}=0$$

From equation (3) we can derive the following:

$$(5)\quad \lambda=\omega_i\left[\sum_j \alpha_j x_j^{\rho}\right]^{1-\frac{1}{\rho}}\alpha_i^{\rho-1}x_i^{1-\rho}$$

Hence, since $\lambda=\lambda$, from equation (5) we then have equation (6) below:

$$(6)\quad x_j=x_i\left(\frac{\alpha_j}{\alpha_i}\right)^{\frac{1}{1-\rho}}\left(\frac{\omega_i}{\omega_j}\right)^{\frac{1}{1-\rho}}$$

We shall now represent the elasticity of substitution by the term σ. It should be clear from equation (6) that the (Allen) substitution elasticity between two inputs x_i and x_j will be $\sigma=1/(1-\rho)$. Hence, we can simplify equation (6) as follows:

$$(7)\quad x_j=x_i\left(\frac{\alpha_j}{\alpha_i}\right)^{\sigma}\left(\frac{\omega_i}{\omega_j}\right)^{\sigma}$$

Recall the production function from equations (1) and (4). Making a substitution of equation (7) into equation (4) leads to the following result.

$$(8)\quad y=\left[x_i^{\rho}\sum_j \alpha_j\left(\frac{\alpha_j}{\alpha_i}\right)^{\rho\sigma}\left(\frac{\omega_i}{\omega_j}\right)^{\rho\sigma}\right]^{1/\rho}$$

We can collect terms in equation (8) that relate to input x_i , yielding the following:

$$(9)\quad y = \left[x_i^{\rho} \alpha_i^{-\rho\sigma} \omega_i^{\rho\sigma} \sum_j \alpha_j^{\ 1+\rho\sigma} \omega_j^{\ -\rho\sigma} \right]^{1/\rho}$$

Yet another set of simplifications can be obtained by manipulating the definition of σ, which yields (1+ρσ)=σ and hence -ρσ=1-σ. We also have 1/ρ=-σ/(1-σ) and 1-(1/ρ)=1/(1-σ). Making these substitutions into equation (9) then yields the following:

$$(10)\quad \begin{aligned} y &= x_i \left(\frac{\omega_i}{\alpha_i} \right)^{\sigma} P^{-\sigma} \\ P &= \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma} \right]^{1-\frac{1}{\rho}} = \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma} \right]^{1/(1-\sigma)} \end{aligned}$$

It turns out that P is the minimum unit cost function for y. To see this, we first rearrange equation (10) to solve for input demand. This yields the following:

$$(11)\quad x_i = y \left(\frac{\alpha_i}{\omega_i} \right)^{\sigma} P^{\sigma}$$

If we set y=1, then the unit cost function from equation (11) can then be seen to be the following:

$$(12)\quad \begin{aligned} c &= \sum_i \omega_i \left(\frac{\alpha_i}{\omega_i} \right)^{\sigma} P^{\sigma} = P^{\sigma} \sum_i \alpha_i^{\sigma} \omega_i^{1-\sigma} = P^{\sigma} \left(\sum_i \alpha_i^{\sigma} \omega_i^{1-\sigma} \right)^{\frac{1-\sigma}{1-\sigma}} = P^{1+\sigma-\sigma} = P \\ P &= \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma} \right]^{1-\frac{1}{\rho}} = \left[\sum_j \alpha_i^{\sigma} \omega_j^{1-\sigma} \right]^{1/(1-\sigma)} \end{aligned}$$

We can also relate total input demand, for cost minimizing firms, to the total expenditure on the good (assuming zero profits), or identically to total expenditure on inputs. If we denote total expenditure as $Y=P^*y$, then total input demand under cost minimization, given total expenditure Y on inputs, can be derived from equation (11).

$$(13)\quad x_i = Y \left(\frac{\alpha_i}{\omega_i} \right)^{\sigma} P^{\sigma-1}$$

To summarize, we have the following:

- Relative input demands are represented by equation (6)
- Total input demand given total output is represented by equation (11)
- Total input demand given total cost is given by equation (13)
- The input values, as we move along an isoquant, are given by equation (11)
- The unit cost function is scale independent, and is represented by P in equation (12)

Note that these results will also hold under alternative market structures as long as we assume profit maximization (which involves cost minimization as part of the sub-problem).